Order-Optimal Permutation Codes in the Generalized Cayley Metric

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Outline

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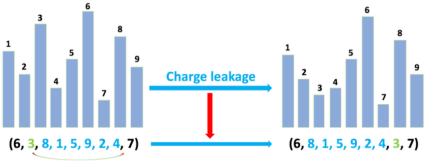

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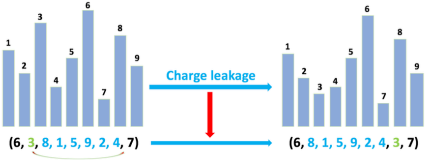

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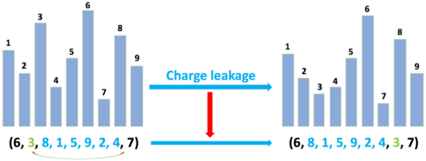

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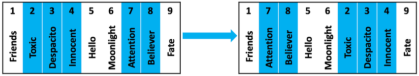
- Cloud storage system: rearrangements of items in multiple folders



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  - Generalized Cayley metric: generalized transposition [5]

  - No restrictions on the lengths and positions of the translocated segments

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Outline

Generalized Cayley Distance

- **Generalized transposition** $\phi(i_1, j_1, i_2, j_2)$:
    - $\phi(i_1, j_1, i_2, j_2) \in \mathbb{S}_N$, $i_1 \leq j_1 < i_2 \leq j_2 \in [N]$, $\mathbb{S}_N$ is the symmetric group of permutations with length N
    - A permutation obtained from swapping the segments $e[i_1, j_1]$ and $e[i_2, j_2]$ in the identity permutation

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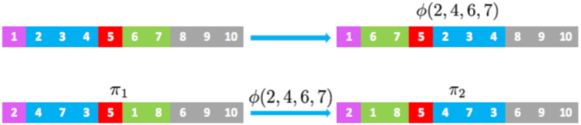

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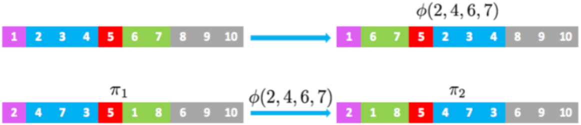

  - $\pi_2 = \pi_1 \circ \phi$
- **Generalized Cayley distance** $d_G(\pi_1, \pi_2)$:
  - The minimum number of generalized transpositions that is needed to obtain the permutation π_2 from π_1,

$$d_G(\pi_1, \pi_2) \triangleq \min_d \{\exists\ \phi_1, \phi_2, \cdots, \phi_d \in \mathbb{T}_N, \text{ s.t., } \pi_2 = \pi_1 \circ \phi_1 \circ \phi_2 \cdots \circ \phi_d\}.$$

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- Exact value of $d_G(\pi_1, \pi_2)$ is hard to compute
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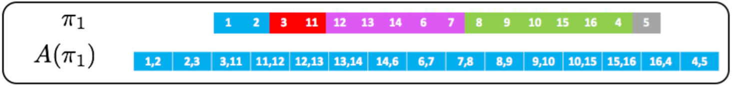
  - Each generalized transposition changes at most 4 elements in the characteristic set (boundaries of the unaltered blocks)



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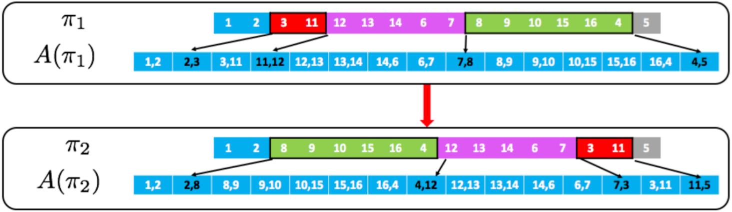

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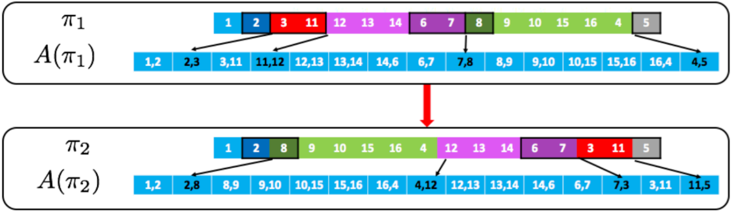

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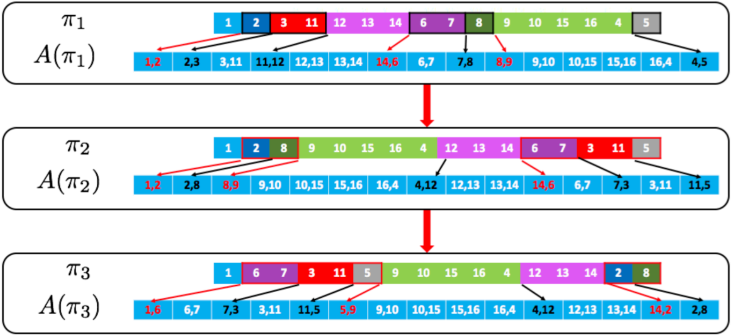

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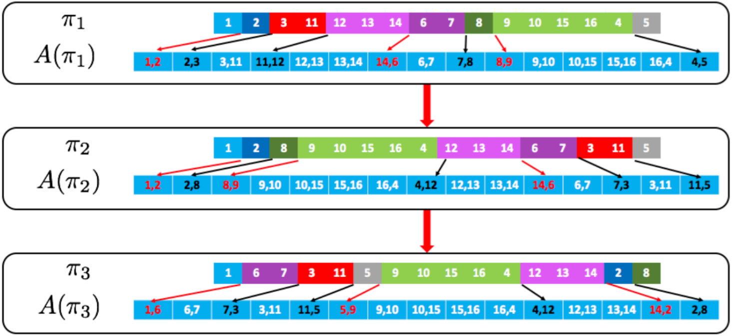

Block Permutation Distance

- **Block permutation distance** $d_B(\pi_1, \pi_2)$:
  - $d_B(\pi_1, \pi_2) = d$ iff $\exists \sigma \in \mathbb{D}_{d+1}$ such that $\forall\ 1 \leq i \leq d$, $\sigma(i+1) \neq \sigma(i) + 1$, $\psi_k = \pi_1 \left[i_{k-1} + 1 : i_k \right]$ for some $0 = i_0 < i_1 \cdots < i_d < i_{d+1} = N$, and $1 \leq k \leq d+1$, such that

$$\pi_1 = \left(\psi_1, \psi_2, \cdots, \psi_{d+1} \right),$$
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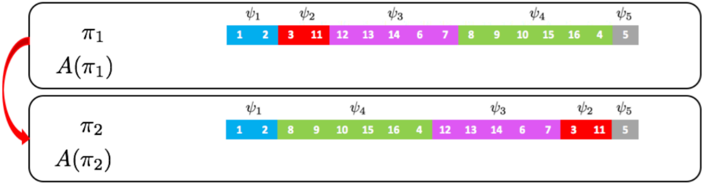

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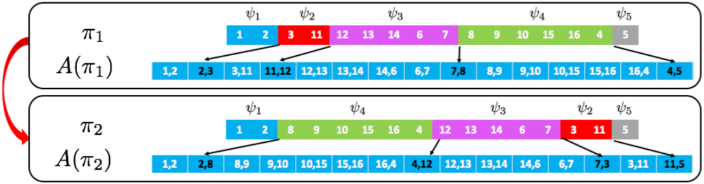

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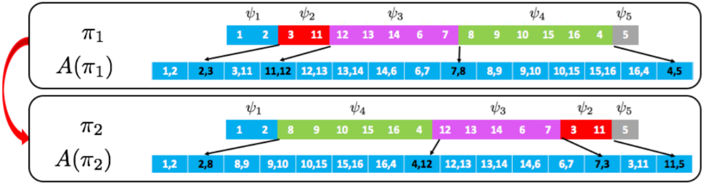

  - $d_B(\pi_1, \pi_2) = \frac{1}{2}|A(\pi_1)\Delta A(\pi_2)|$
- Metric embedding:

$$d_G(\pi_1, \pi_2) \le d_B(\pi_1, \pi_2) \le 4d_G(\pi_1, \pi_2)$$

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- Order-optimal $4t$-block permutation codes are order-optimal t-generalized Cayley codes

Theorem

The optimal rates satisfy the following inequalites,

$$1 - c_1 \cdot \frac{2t+1}{N} \leq R_{B,opt}(N,t) \leq 1 - \frac{t}{N},$$
$$1 - c_1 \cdot \frac{8t+1}{N} \leq R_{G,opt}(N,t) \leq 1 - c_2 \cdot \frac{4t}{N},$$

for fixed t and sufficiently large N, where $c_1 = 1 + \frac{2\log e}{\log N}$, $c_2 = 1 - \frac{3(\log t+1)}{4(\log N-1)}$.

Outline

Key Idea in Encoding Scheme

π_1	1	2	3	11	12	13	14	6	7	8	9	10	15	16	4	5
$A(\pi_1)$																
$g(\pi_1)$																

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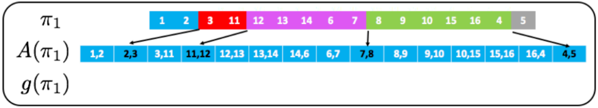

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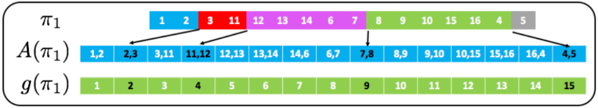

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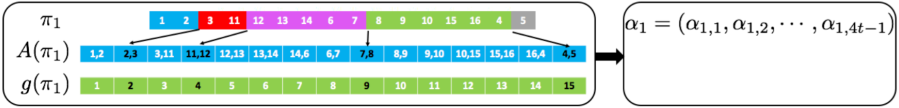

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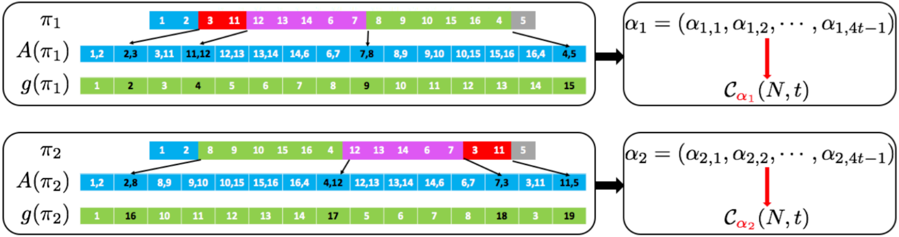

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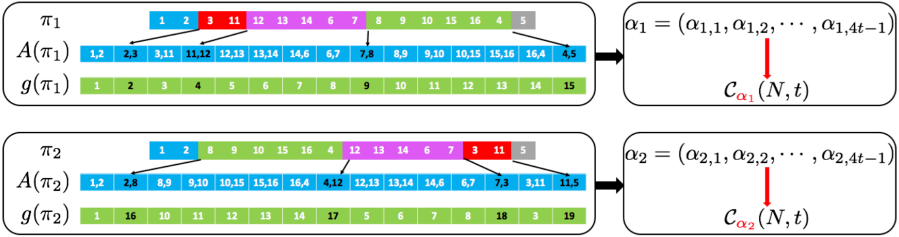

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Note: $\mathcal{C}_\alpha(N, t)$ with the maximum cardinality is order-optimal

Auxiliary Bound Results

Theorem

For all $B_1, B_2 \subset \mathbb{F}_q$, if $h_t(B_1) = h_t(B_2)$, then $|B_1 \Delta B_2| > 4t$.

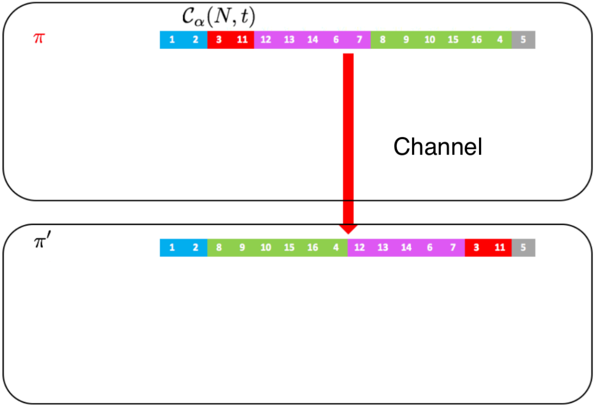
Proof.

If $|B_1 \Delta B_2| \leq 4t$, then $B_1 \setminus B_2 = \{x_1, x_2, \cdots, x_k\}$, $B_2 \setminus B_1 = \{x_{k+1}, x_{k+2}, \cdots, x_{2k}\}$, $k \leq 2t$.

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{2k} \\ x_1^2 & x_2^2 & \cdots & x_{2k}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{2d-1} & x_2^{2d-1} & \cdots & x_{2k}^{2d-1} \end{pmatrix} \mathbf{y} = \mathbf{0},$$

where $\mathbf{y} = [y_1, y_2, \cdots, y_{2k}]^T$, $y_i = 1(i \leq k)$, $y_i = -1(i > k)$.
The Vandermonde matrix has determinant $0 \implies \exists i, j$ such that $x_i = x_j$, contradiction! □

Key Steps in Decoding Algorithm



Channel: Receiver receives π' when sender sends π, $d_B(\pi, \pi') \leq t$

Key Steps in Decoding Algorithm

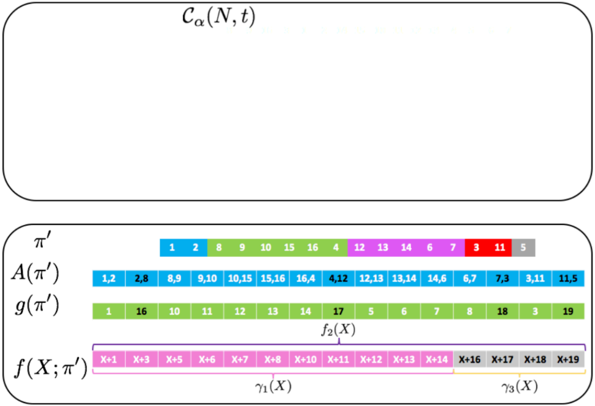
$\mathcal{C}_\alpha(N,t)$

π'		1	2	8	9	10	15	16	4	12	13	14	6	7	3	11	5
$A(\pi')$	1,2	2,8	8,9	9,10	10,15	15,16	16,4	4,12	12,13	13,14	14,6	6,7	7,3	3,11	11,5	
$g(\pi')$	1	16	10	11	12	13	14	17	5	6	7	8	18	3	19	

Channel: Receiver receives π' when sender sends π, $d_B(\pi, \pi') \leq t$

Step 1: Compute $A(\pi')$, $g(\pi')$ and $f_2 = (X; \pi')$ from π'

Key Steps in Decoding Algorithm



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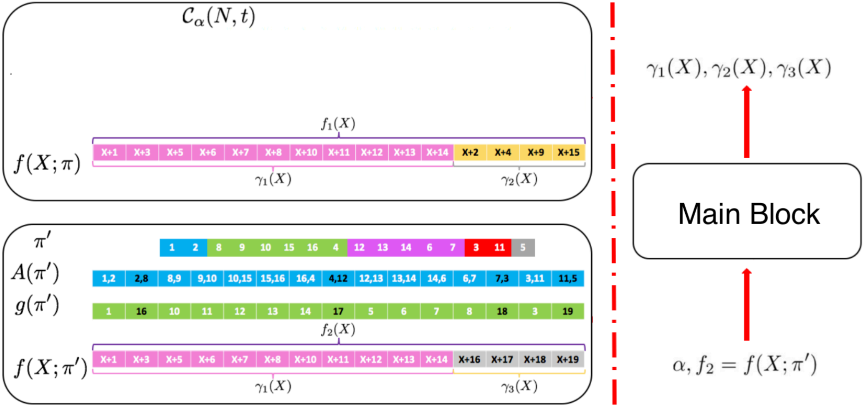

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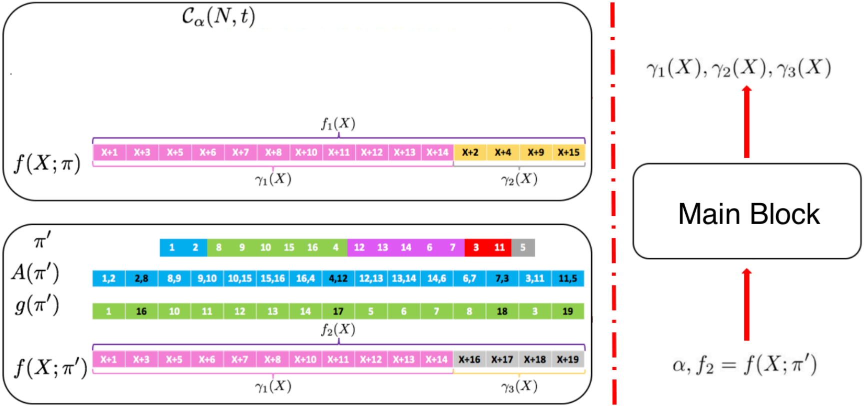
f_2 provides incomplete information about the roots of f_1

α provides complete information about the $4t - 1$ coefficients of f_1

Key Steps in Decoding Algorithm



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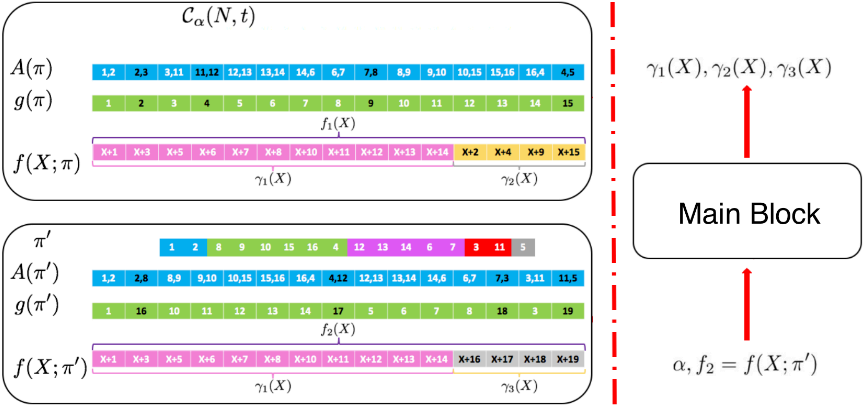

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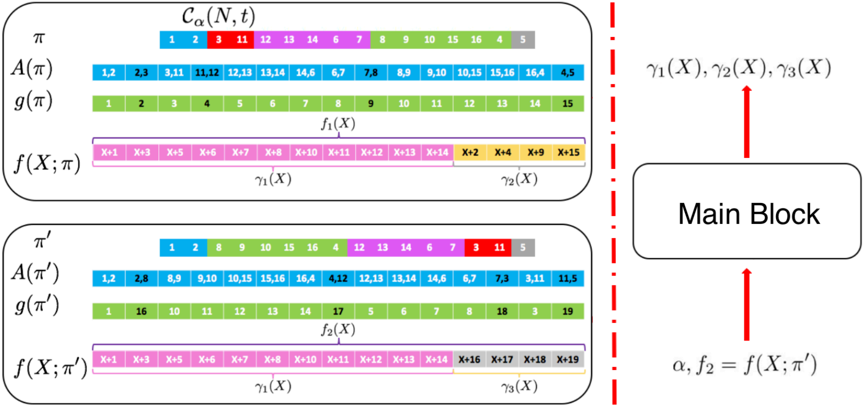

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Main Block

- $(X^{t-k}\gamma_3, X^{t-k}\gamma_2)$ is a solution to $h_1 \circ f_1 = h_2 \circ f_2$, $\deg h_1 = \deg h_2 = t$
  - Any solution (h_1, h_2) is sufficient for computing γ_2, γ_3: $\gamma_1 = gcd(h_1, h_2)$, $\gamma_3 = \frac{h_1}{\gamma_1}$, $\gamma_2 = \frac{h_2}{\gamma_1}$;
- The first $4t$ constraints of the coefficients for h_1 cot $f_1 = h_2 \cdot f_2$ is $\mathbf{Ac} = \mathbf{b}$
  - The coefficient of $X^{N+t-k-1}$ in f: a_k, $(a_1, \cdots, a_{4t-1})$ is known from Newton's Identities
  - The coefficient of X^{t-k} in h: c_k
  - We can compute the coefficients of h_1, h_2 from the solution of $\mathbf{Ac} = \mathbf{b}$

$$\underbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots & a'_1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 & \vdots & \vdots & \ddots & 0 \\ a_{t-1} & a_{t-2} & \cdots & 1 & a'_{t-1} & a'_{t-2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{4t-2} & a_{4t-3} & \cdots & a_{3t-1} & a'_{4t-2} & a'_{4t-3} & \cdots & a'_{3t-1} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} c_1 \\ \vdots \\ c_t \\ -c'_1 \\ \vdots \\ -c'_t \end{pmatrix}}_{\mathbf{c}} = \underbrace{\begin{pmatrix} a'_1 - a_1 \\ \vdots \\ a'_{4t-1} - a_{4t-1} \end{pmatrix}}_{\mathbf{b}}$$

Rate Comparison with Interleaving Based Codes

Lemma

Let $R_G(N,t)$, $R_{\rho_g C}(N,t)$ be the rate of our proposed code and the existing interleaving-based code, respectively. Then $R_G(N,t) > R_{\rho_g C}(N,t)$ when $t < \frac{N}{(16\log N+8)}$ for sufficiently large N.

Proof.

We know from previous discussion and [a] that

$$\begin{aligned} R_{\rho_g C}(N,t) &< 1 - \frac{2N + \mathcal{O}\left((\log N)^2\right)}{N\log N - (\log e)N + \frac{1}{2}\log N} \sim 1 - \frac{2}{\log N}, \\ R_G(N,t) &> 1 - \frac{32t\log N + 16t}{N\log N - (\log e)N + \frac{1}{2}\log N} \sim 1 - \frac{32t}{N}, \end{aligned} \quad (1)$$

$R_G(N,t) - R_{\rho_g C}(N,t) > 0$ when $t < \frac{N}{(16\log N+8)}$ for sufficiently large N. □

[a] R. Gabrys et al. "Codes Correcting Erasures and Deletions for Rank Modulation". In: *IEEE Trans. Inf. Theory* 62 (Jan. 2016), pp. 136–150.

Outline

Extension

- Problems in the previous construction

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  - Not explicitly constructive

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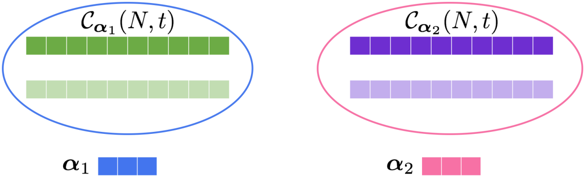
  - Extended work submitted to IEEE Trans. Information Theory, also available at arxiv: https://arxiv.org/abs/1803.04314

Systematic Codes in the Generalized Cayley Metric



- Main idea: insert k elements $[N+1:N+k]$ into the length N permutations at positions decided by their parity check sums

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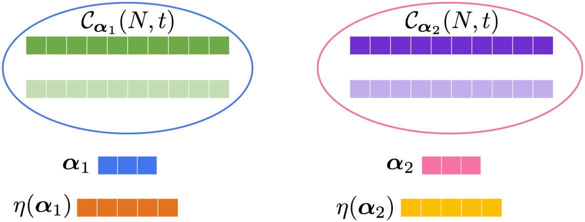

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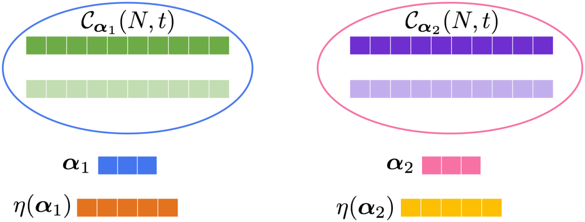

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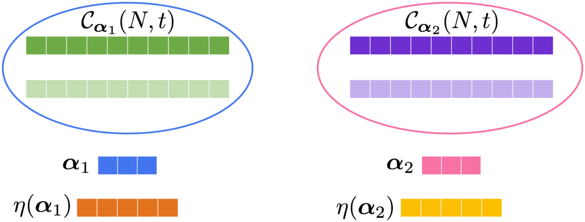

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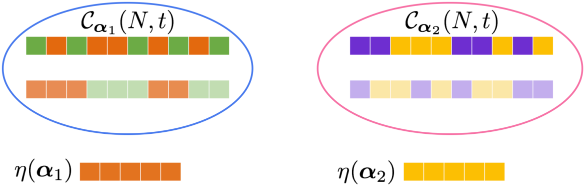

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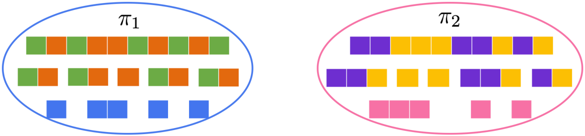

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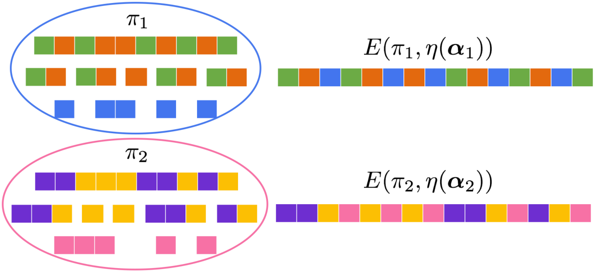

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  - Insert $N+i$, $1 \le i \le k$ sequentially after the element in π identical to the i-th element in $\eta(\boldsymbol{\alpha})$
  - New permutations also have distance at least $2t+1$

Extension of Permutations (Definition 5)

- **Extension** of π at the **extension point** s, $\pi \in \mathbb{S}_N$, $s \in [N]$:
  $E(\pi, s) \triangleq (\pi_1, \pi_2, \cdots, \pi_k = s, N+1, \pi_{k+1}, \cdots, \pi_N)$
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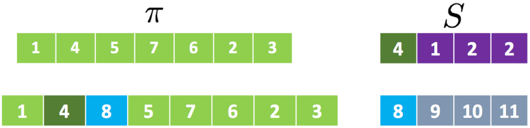

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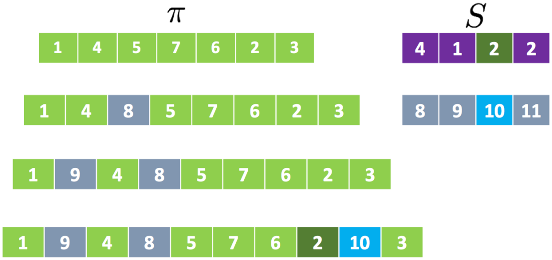

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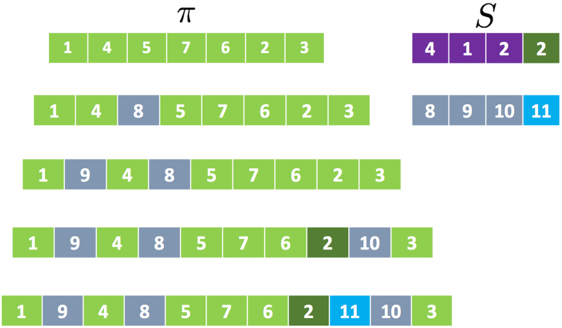

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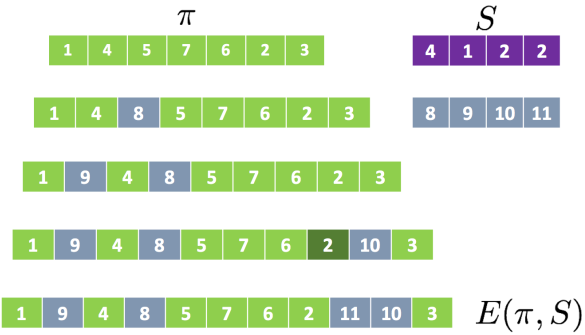

Jump Points of Extensions

- s_1 is **Jump point** of $\sigma_1 = E(\pi_1, s_1)$ with respsect to $\sigma_2 = E(\pi_2, s_2)$ if (suppose $\pi_{1,k_1} = s_1$ and $\pi_{2,k_2} = s_2$)

  Case 1 $k_1 = N$ or $k_2 = N$;

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    - Case 2 $((A(\pi_1) \setminus A(\pi_2)) \setminus \{(s_1, \pi_{1,k_1+1})\}) \cup \{(s_1, N+1), (N+1, \pi_{1,k_1+1})\} \subset A(\sigma_1) \setminus A(\sigma_2)$

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Step 1 Given a t-auxiliary set $\mathcal{A}(N, K, t)$ with cardinality that is no less than q^{4t-1}

Step 2 Find an injection $\varphi: \ q^{4t-1} \to \mathcal{A}(N, K, t)$, where q is a prime number such that $N^2 - N < q < 2(N^2 - N)$

Step 3 Compute the set $\mathcal{C}_B^{\mathrm{sys}}(N, K, t) = \{E(\pi, \varphi \circ \alpha(\pi)) | \pi \in \mathbb{S}_N\}$

Theorem 4 $\mathcal{C}_B^{\mathrm{sys}}(N, K, t)$ is a systematic t-block permutation code

For any $\pi_1 \neq \pi_2$, Let $\sigma_1 = E(\pi_1, \varphi(\alpha_1)), \sigma_2 = E(\pi_2, \varphi(\alpha_2))$

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Note Only need to construct t-auxiliary set $\mathcal{A}(N, K, t)$ with cardinality that is no less than q^{4t-1} (will introduce later)

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Step 4 $d_B(\pi, \pi') \leq d_B(\sigma, \sigma') \leq t$, decode π from π' and $\alpha(\pi)$ using **Theorem 3**

Construction: t-Auxiliary Set

Lemma 14 For all $k, N \in \mathbb{N}^*$, $k > 3$, $N > k^2$, consider an arbitrary subset $Y \subset [k]$, where $|Y| = M < k$, $Y = \{i_1, i_2, \cdots, i_M\}$, then $\mathrm{LCM}(N + i_1, N + i_2, \cdots, N + i_M) > N^{M - \frac{k}{2}}$

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Lemma 16 Code constructed by **Theorem 4** using $\mathcal{A}(N, 56t, t)$ is systematic and order-optimal

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- Future work
  - Binary codes that corrects generalized transposition error (has potential in DNA storage)

Thank you!